Lecture 17

- **Electric conduction**
- **Electrons' motion in magnetic field**
- **Electrons' thermal conductivity**

Brief review

In solid state physics, we do not think about electrons zipping around randomly in real space. This is because if we define electrons as plane waves characterized by a **momentum** state $\psi \sim e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$. When described in **momentum space** these electrons form an organized structure called a Fermi sphere, which is a sphere in k-space divided into tiny 'boxes' of dimension $\left(\frac{2\pi}{L}\right)^3$ which can each hold one spin up and one spin down electron. Only electrons very close to the boundary of the sphere do anything—all the others are inert (in contrast to the real-space picture where all valence electrons contribute to metallic conduction). The electrons close to the boundary have

(energy): $\epsilon = \epsilon_F = \frac{\hbar^2 k_F^2}{2m}$

(momentum): $k = k_F$

(velocity) $v = v_F = \hbar k_F / m$

For this lecture, we will not explicitly use this formalism very much, and will revert to a more classical picture of electric conduction. The most important part of this portion of the textbook are the concepts and language, which are used in modern solid state physics research, but a more rigorous description of electrical conduction, in terms of the Fermi sphere construction, will be left to a later course.

Electrical conductivity (semi-classical treatment)

The Fermi sphere structure of electrons in a metal, with a hierarchy of energy states, provides a much more organized way of understanding electrical conduction than the real-space picture of electrons haphazardly zipping around and bumping into things.

The momentum of a free electron is related to its wavevector by

$$m\boldsymbol{v} = \hbar\boldsymbol{k}$$

In an electric field **R** and magnetic field **B**, the force on an electron (charge e) is given by:

$$\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} = \hbar\frac{d\boldsymbol{k}}{dt} = -e(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B})$$

We set B=0 for now.

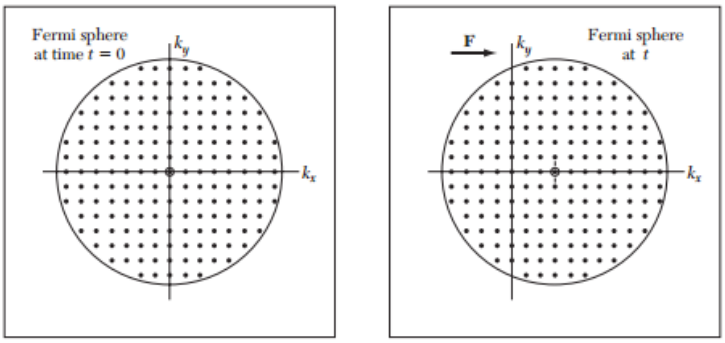
In the absence of collisions, the Fermi sphere will be accelerated by an electric field as a unit. If a force **F**=-e**E** is applied at t=0 to an electron gas, an electron with initial wavevector **k**(0) will end up at a final wavevector **k**(t)

$$\boldsymbol{k}(t) - \boldsymbol{k}(0) = -e\boldsymbol{E}t/\hbar$$

This statement applies to every electron in the Fermi sea without regards to the specific momentum or energy that electron has, so a Fermi sphere centered at **k**=0 at t=0 will have its center displaced by

$$\delta \boldsymbol{k} = -e\boldsymbol{E}t/\hbar$$

This also corresponds to a velocity kick $\delta \boldsymbol{v} = \hbar\delta \boldsymbol{k}/m$ (found by replacing derivatives in equation of motion by infinitesimal changes δk and δv)



The Fermi sphere does not accelerate indefinitely, because electrons eventually do scatter with lattice imperfections, impurities, or phonons. This characteristic scattering time is called τ, which gives a 'steady state' value of $\delta \boldsymbol{k} = -\frac{e\boldsymbol{E}\tau}{\hbar} = m\delta \boldsymbol{v}/\hbar$

Thus, the incremental velocity imparted to electrons by the applied electric field is $\delta \boldsymbol{v} = \frac{\delta \boldsymbol{k}}{m} = -e\boldsymbol{E}\tau/m$

If in a steady electric field, there are n electrons per unit volume, the current density (**j**) is given by

$$\boldsymbol{j} = ne\delta \boldsymbol{v} = ne^2\tau \boldsymbol{E}/m$$

This is a generalized version of Ohm's law, because **j** is related to the current I, and electric field is related to a voltage or potential difference.

The electrical conductivity is defined by $\sigma = \frac{ne^2\tau}{m}$

And the resistivity (ρ) is defined as the inverse of conductivity

$$\rho = \frac{m}{ne^2\tau}$$

Resistivity is related to resistance (R) via a materials geometry, so resistivity is considered to be a more fundamental quantity because it does not depend on geometry

$$R = \frac{\rho \ell}{A}$$

Where ℓ is the length of the specimen, and A is the cross sectional area.

What is the physical origin of a finite τ?

The derivation above stipulates that electrons **scatter**—bump into something and lose their momentum information—every interval τ, which in real materials tends to be on the order of 10^{-14}s, depending on temperature.

- At room temperature, phonons provide the primary scattering mechanism for electrons. To be clear, a perfect lattice will not scatter electrons and will not contribute to resistivity, but at higher temperature, a crystal lattice becomes increasingly 'imperfect' (because of increased atomic vibrations) which allows increased scattering off the lattice. Or, if one views phonons as emergent particles with a certain energy and momentum, electrons scatter off these 'particles' such that the total energy and momentum is conserved. This type of scattering happens every time interval τ_L, which depends on temperature
- At cryogenic temperature, electrons primarily scatter off impurities and other permanent defects in the crystalline lattice. This type of scattering happens every time interval τ_i, and is independent of temperature.

The scattering frequency (inverse of scattering time) is given by adding up scattering frequencies from each contribution:

$$\frac{1}{\tau} = \frac{1}{\tau_L} + \frac{1}{\tau_i}$$

This also implies that the contribution to resistivity from each type of scattering adds up linearly

$$\rho = \rho_L + \rho_i$$

Example: A copper sample has a residual resistivity (resistivity in the limit of T=0) of 1.7e-2 $\mu\Omega\ cm$. Find the impurity concentration.

Solution:

At zero temperature, only impurities contribute to resistivity

$$\rho = m/ne^2\tau$$

Solve for τ.

n is the electron concentration, and copper has 1 valence electron per atom. Copper forms an FCC structure (4 atoms per cubic cell) with a unit cell dimension of 3.61e-10m. Thus, $n = 8.5x10^{28}m^{-3}$

to solve for τ, first change the units of ρ. $\rho = 1.7 \times 10^{-10}\Omega\ \mathrm{m}$

$$\tau = \frac{m}{ne^2\rho} = 2.46 \times 10^{-12}s$$

This can be used to solve for an average distance (ℓ) between collisions using

$$\ell = v_F\tau$$

where v_F is the Fermi velocity

$$v_F = \left(\frac{\hbar}{m}\right)\left(\frac{3\pi^2 N}{V}\right)^{1/3} = 1.6 \times 10^6 m/s$$

$$\ell = 3.9\mu m$$

Thus, given the T=0 resistivity of this specimen, the average spacing between impurities is $3.9\mu m$ which means an electron would travel on average $\frac{(3.9\times10^{-6})}{(3.61\times10^{-10})} = 12{,}341$ unit cells before encountering an impurity

Electrons' motion in a magnetic field

In an electric field **R** and magnetic field **B**, the force on an electron (charge e) is given by:

$$\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} = \hbar\frac{d\boldsymbol{k}}{dt} = -e(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B})$$

Again, we consider displacing the Fermi sphere by a momentum $\delta\boldsymbol{k}$ such that

$$m\boldsymbol{v} = \hbar\delta\boldsymbol{k}$$

Where **v** is the incremental velocity kick that all electrons get.

We express acceleration in a slightly different way than we did previously to write expressions for motion in electric and magnetic field applied simultaneously (previously, we dropped the first term on the left because in the steady state, time derivatives are zero, but this notation is being introduced because it is needed to study time-varying fields, like in your homework):

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)\boldsymbol{v} = -e(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B})$$

A special case of this problem arises when the magnetic field is applied along the z axis ($\boldsymbol{B} = B\hat{\boldsymbol{z}}$):

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)v_{\boldsymbol{x}} = -e(E_x + \frac{Bv_y}{c})$$

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)v_{\boldsymbol{y}} = -e(E_y - \frac{Bv_x}{c})$$

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)v_{\boldsymbol{z}} = -e(E_z + 0)$$

In steady state, the time derivatives are zero, so the first terms on the left side disappear. These equations then become:

$$v_x = -\frac{e\tau}{m}E_x - \omega_c\tau v_y$$

$$v_y = -\frac{e\tau}{m}E_y + \omega_c\tau v_x$$

$$v_z = -\frac{e\tau}{m}E_z$$

Where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. The cyclotron frequency describes the frequency of electrons' circular motion in a perpendicular magnetic field. It is notable independent of the electron's velocity or the spatial size of the circular orbit, and it only depends on a particle's charge-to-mass ratio.

Hall effect

The hall effect refers to a transverse voltage that develops when a current flows across a sample at the same time that a magnetic field is applied in the perpendicular direction. It is a very important characterization tool for assessing the number of charge carriers and their charge.

In general, the transverse electric field will be in the direction $\boldsymbol{j} \times \boldsymbol{B}$**,** and customarily, the current (**j)** direction is set perpendicular to the magnetic field direction.

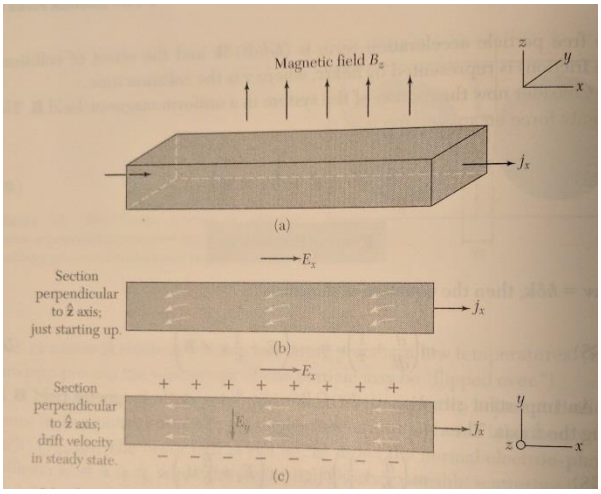
We consider a specific case where $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ and $\boldsymbol{j} = j\hat{\boldsymbol{x}}$

When electrons flow with a velocity v_x perpendicular to the direction of the magnetic field, they will feel a force in the $\boldsymbol{v} \times \boldsymbol{B}$ direction, which is in the -y direction. Thus, there will be an accumulation of negative charges on the –y side of the sample, leading to an electric field in the –y direction. Note that this electric field will tend to deflect directions in the opposite direction from the magnetic field, so a steady state situation is reached where the Lorentz force from the magnetic field perfectly balances the force from the electric field.

To write this more quantitatively:

Use: $v_y = -\frac{e\tau}{m}E_y + \omega_c \tau v_x$ and set $v_y = 0$ to reflect the steady state situation when there is no more y-deflection

$$E_y = -\omega_c \tau E_x = -\frac{eB\tau}{mc}E_x$$



A note about signs:

Electric current is defined as the flow of **positive** charges, so the electron velocity is in the opposite direction to the current (in this case, -x) [thus, $\boldsymbol{v} \times \boldsymbol{B} = vB\hat{\boldsymbol{y}}$]

The negative sign is explicitly included in the definition of force (that an electron feels in a perpendicular magnetic field), so electrons will accelerate to the –y side of the sample

The direction of the electric field is defined as the direction of the force that a **positive** test charge will feel, so electric field direction always points from positive to negative charges (towards –y in this case)

A **hall coefficient** is defined as

$$R_H = \frac{E_y}{j_x B}$$

Use $j_x = \frac{ne^2\tau E_x}{m}$ and $E_y = -\frac{eB\tau}{m}E_x$ to evaluate

$$R_H = \frac{-eB\tau/mE_x}{ne^2\tau E_x B/mc} = -\frac{1}{nec}$$

The two free parameters here are n (the electron concentration) and the sign of e. In some metals, the dominant charge carriers are electrons, and in other metals, it is the voids left behind by electrons, which are called holes (and are mathematically equivalent to positrons, physical particles which are positively charged electrons. The sign of the hall coefficient distinguishes between those two cases. Additionally, the number of mobile charge carriers in a metal might be different from the number of valence electrons you think you have, and hall coefficient measurements can detect that too. In some carriers, both electrons and holes can be charge carriers, each with different densities, and in those cases, interpretation of the hall coefficient can be tricky.

Thermal conductivity of metals

In Ch5 we considered the thermal conductivity if heat could **only** be carried by phonons. In metals, heat can be carried by electrons too.

For phonons, thermal conductivity is given by $K = \frac{1}{3}Cv\ell$ and we can identify analogous quantities for metals.

For phonons, v is the sound velocity—the group velocity that acoustic phonons follow. For electrons, the equivalent quantity is the **Fermi velocity** (v_F)—the group velocity of electrons at the Fermi energy (most electrons in a metal are inert, except for those that happen to have energy within $\sim k_B T$ of the Fermi energy.

C is the heat capacity per unit volume, and earlier in this chapter we calculated heat capacity for electrons

$$C_{el} = \frac{1}{2}\pi^2 N k_B T/T_F$$

$$T_F = \frac{\epsilon_F}{k_B} = \frac{\frac{1}{2}mv_F^2}{k_B}$$

$$C_{el} = \frac{\pi^2 N k_B^2 T}{mv_F^2}$$

This is the total heat capacity, and we need to divide by a factor of V to get the heat capacity per volume

$$C = \frac{\pi^2 n k_B^2 T}{mv_F^2}$$

Plugging this in to the expression for the thermal conductivity coefficient:

$$K_{el} = \frac{\pi^2 n_{electrons} k_B^2 T}{3 m v_F^2} v_F \ell$$

Compare this to the phonon thermal conductivity at low temperature (when ℓ does not depend on temperature)

$$K_{ph} = \frac{4\pi^4}{5} n_{primitive\ cells} k_B \left(\frac{T}{\theta}\right)^3 v_s \ell$$

And the phonon thermal conductivity at high temperature when C does not depend on T, but $\ell \propto \sim 1/T$

$$K_{ph} \propto n_{primitive\ cells} k_B v_s / T$$

Or the high-temperature phonon thermal conductivity in the "dirty limit" when impurities set ℓ (average impurity distance is given the symbol D), rather than phonon-phonon scattering setting ℓ

$$K_{ph} = 3 n_{primitive\ cells} k_B D$$

We can further express the electronic thermal conductivity in terms of the mean scattering time $\tau = \ell / v_F$

$$K_{el} = \frac{\pi^2 n_{electrons} k_B^2 T \tau}{3m}$$

It turns out that in pure/clean metals, electrons are more effective at transporting heat than phonons, but in metals with many impurities, the two types of thermal conductivity are comparable.

$$K_{total} = K_{electron} + K_{phonon}$$

Wiedemann-Franz law

Since the same electrons carry both electric current and heat, there is an expected ratio between thermal conductivity and electrical conductivity:

$$\frac{K}{\sigma} = \frac{\pi^2 k_B^2 T n \tau / 3m}{n e^2 \tau / m} = \frac{\pi^2}{3} \left(\frac{k_B}{e}\right)^2 T$$

Interestingly, materials' dependent parameters such as τ , n, and m drop out of this ratio.

The Lorentz number L is defined as

$$L = \frac{K}{\sigma T} = \frac{\pi^2}{3} \left(\frac{k_B}{e}\right)^2 = 2.45 \times 10^{-8} Watt\ \Omega / \mathrm{K}^2$$

Most simple metals have values of L roughly in this range (see table 6.5 in textbook)

The Wiedemann-Franz law is a very useful metric in contemporary research for assessing how much an exotic material behaves like a 'simple' or 'textbook' metal which is expected to follow W-F law. Deviation from W-F law in temperature regimes where it should apply are used as evidence that a given material has 'abnormal' behavior.

Example: thermoelectrics

Thermoelectrics are materials that can convert waste heat into electricity (and vis versa: use a voltage to affect a temperature change), and they are defined by a figure of merit ZT. The larger ZT, the better, but most of the best thermoelectrics have ZT~1-2.

$$ZT = \frac{\sigma S^2 T}{\mathrm{K}}$$

Where S is the Seebeck coefficient. This is a materials property which describes the degree to which a temperature gradient produces an electric potential: $S = -\Delta V/\Delta T$

σ is the electrical conductivity and K is the thermal conductivity. According to the equation above, one can increase ZT for a given material (fixed S) by increasing σ or decreasing K.

But as we learned in the previous section, **electrons** carry both charge and heat, and there is a specific ratio between the two, so there is no way to simultaneously raise one and lower the other.

A trick that people often employ is manipulating the **phonon** thermal conductivity.

$$K_{total} = K_{electron} + K_{phonon}$$

For example, by creating nanostructured materials (small D; phonons scatter off the boundaries and have difficulty conducting heat), people can suppress the phonon contribution to thermal conductivity

$$K_{ph} = 3n_{primitive\ cells} k_B D$$

Without hurting electrical conductivity too much.

The other option to make a small D determine phonon thermal conductivity is to put in a bunch of impurities, but that can negatively affect electrical conductivity (by producing a small τ) and degrade the figure of merit.