Superconductivity

This short course is intended for students who have an undergraduate-level understanding of solid state physics and prerequisite content, are delving into (probably experimental) research on superconductivity, and do not have a superconductivity course at their university. This content primarily draws from the following sources:

Kittel solid state physics Ch10

Superconductivity, superfluids, and condensates by James Annett

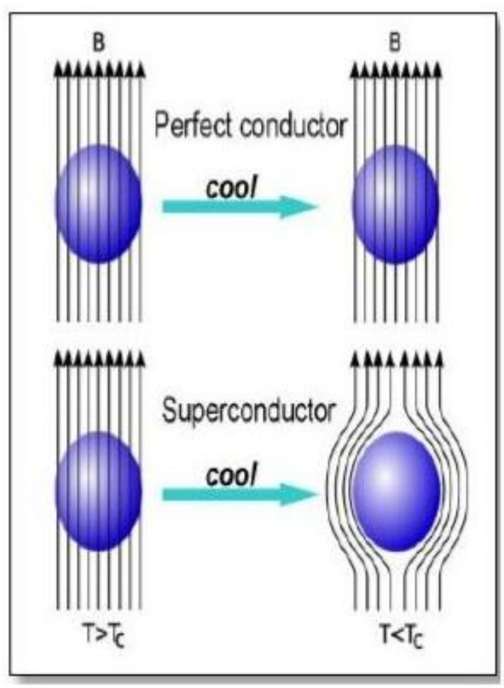
Introduction to superconductivity by Michael Tinkham

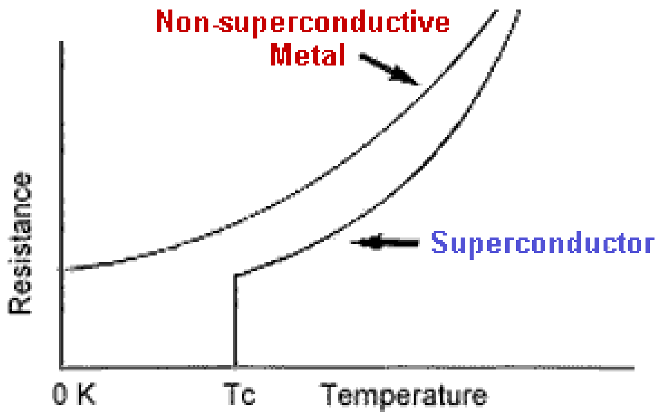
The outline of this lecture is

- Experimental survey
- London equations
- Coherence length
- Brief overview of Ginzburg-Landau
- Brief overview of BCS theory
- Josephson junctions
- Superconducting phase, pairing symmetries, and different types of superconductors

Survey of experimental properties

Superconductors are characterized by two experimental observables: zero resistivity that onsets suddenly at a transition temperature (Tc) and an expulsion of magnetic field (Meissner effect). Tc's in excess of 200K have been reported and confirmed, but superconductivity is generally considered a low temperature phenomenon, with 'typical' Tc's on the order of 1-10K. Superconductivity is a macroscopic quantum state that forms the ground state of some but not all materials. Many elements on the periodic table become superconductors at low temperature, some only under high pressure.





Notably a superconductor is different from a hypothetical perfect conductor.

- A metal completely free of defects at 0K (something that doesn't exist) will be a perfect conductor because there is nothing to scatter electrons, but it will reach the zero resistance state gradually, not suddenly as in a superconductor
- A perfect conductor will want to maintain its present magnetization in the presence of a magnetic field, whereas a superconductor will expel magnetic field always
  - If a magnetic field is turned on, both a superconductor and a perfect conductor will set up currents to expel it
  - If a magnetic field is already turned on and a superconductor is cooled below its transition temperature, it will suddenly expel its magnetic field, while a perfect conductor

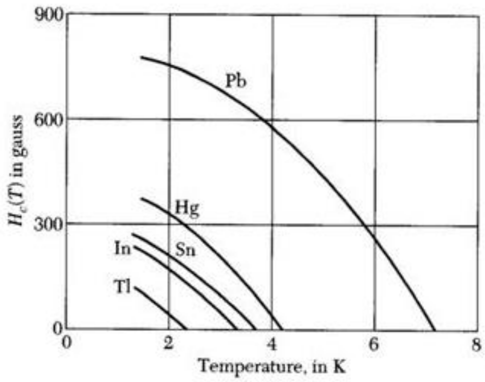
There are some superconductors that scientists 'understand' (microscopic mechanism of superconductivity is explained) and others that are still being researched for the purpose of uncovering the mechanism of their behavior. Many of the topics in this chapter apply to some superconductors, but a few apply only to superconductors that are well understood.

Current applications of superconductors include

- MRI machines (to produce a large magnetic field in a solenoid with no heating)
- Detectors for astrophysics (superconductor kept exactly at transition temperature, and any impinging particle will heat it up slightly and produce finite resistivity)
- Definition of the volt (using josephson junctions, a superconducting device we will discuss in later lectures
- Sensitive detectors of small magnetic fields (also using josephson junctions)
- The most mature quantum computing technology so far (e.g. d-wave)

Properties shared by all superconductors

Destruction of superconductivity by magnetic field



As a note, the figures used when discussing magnetic fields in a superconductor come from different sources and use different letters to describe related concepts. To review:

B=magnetic flux density; units =Tesla; this is what charged particles respond to via $F = q\boldsymbol{v} \times \boldsymbol{B}$
H=magnetic field; units= A/m or Oersted; this one takes the medium into account

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}$$

In vacuum, $\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0}$ in SI units, and in CGS units, **H=B**

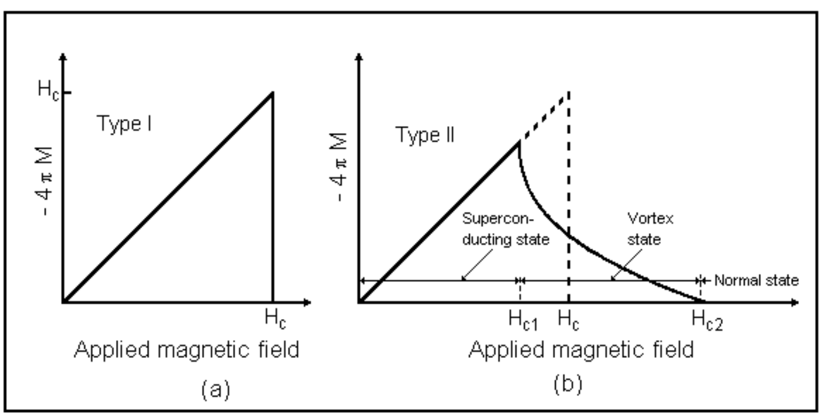
M is the magnetization and μ_0 is the vacuum permeability. In vacuum B and H are proportional and in the same direction, but in a medium they do not have to be. Note B and H are sometimes both called magnetic field.

A sufficiently large magnetic field will destroy superconductivity, and for now this critical magnetic field is called H_c (subtleties coming next). H_c is a function of temperature, and is highest at zero temperature (when superconductivity is strongest) and zero at Tc. Generally, higher Tc corresponds to higher $H_c(T = 0)$, but not always.

Meissner effect

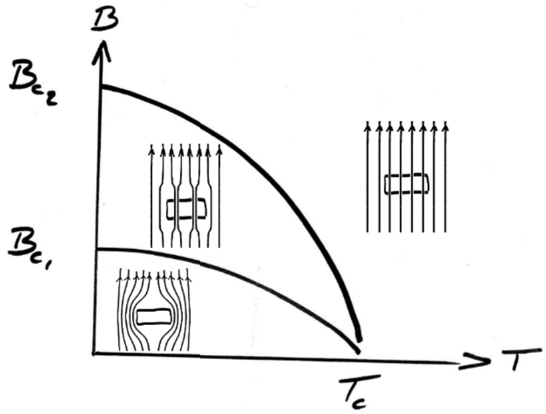
The Meissner effect is the property that a superconductor expels magnetic field in the superconducting state (CGS units)

$$B = B_a + 4\pi M = 0$$



B_a is the applied magnetic field, and M is the magnetization of the specimen. Note that in CGS units, $H_c \equiv B_{ac}$ (subscript c=critical, subscript a=applied), which is why the figures in this section switch back and forth. Note that this relation is only exact in specimens that have a specific shape so that the demagnetizing field is irrelevant.

For some superconductors, the relationship above is obeyed for all magnetic fields, up until the magnetic field is so large that it kills superconductivity (H_c, where c=critical, more on that later). These are called **type-I superconductors**. For other types of superconductors (**type-II superconductors**), the relationship above is obeyed up until an intermediate magnetic field, H_{c1}, at which point the superconductor expels some but not all of its magnetic field. Superconductivity is not completely destroyed until a higher magnetic field, H_{c2}. For a type-II superconductor in the vortex state ($H_a > H_{c1}$), magnetic field is allowed to enter via small filaments called 'vortices'. Each filament contains one flux quantum of magnetic flux ($\frac{h}{2e}$ in SI units or $\frac{hc}{2e}$ in CGS). It is called a vortex because the magnetic core is surrounded by a circulating supercurrent (a superconducting current), like a tiny tornado.



Heat Capacity

Superconductivity constitutes a thermodynamic phase transition, which means that there is a discontinuity in free energy (U) or some derivative thereof. Evidence of these discontinuities is measured via heat capacity, and heat capacity (at constant volume) can be connected to entropy via the following relations:

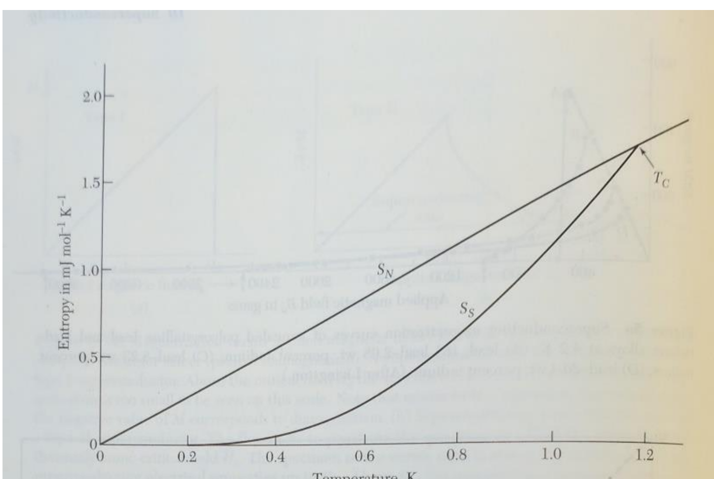


Figure 6 Entropy S of aluminum in the normal and superconducting states as a function of the temperature. The entropy is lower in the superconducting state because the electrons are more ordered here than in the normal state. At any temperature below the critical temperature T_c the specimen can be put in the normal state by application of a magnetic field stronger than the critical field.

$$C_v(T) \equiv \left(\frac{\partial U}{\partial T}\right)_{V,N}$$

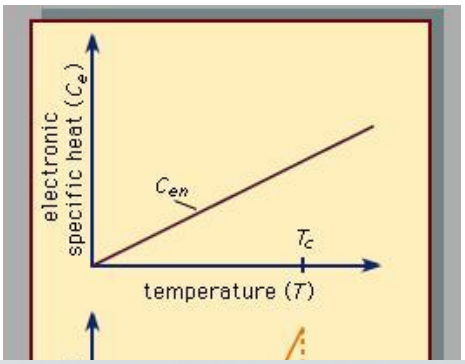
$$\left(\frac{\partial U}{\partial T}\right)_{V,N} = \left(\frac{\partial U}{\partial S}\right)_{V,N}\left(\frac{\partial S}{\partial T}\right)_{V,N}$$

$$= T\left(\frac{\partial S}{\partial T}\right)_{V,N}$$

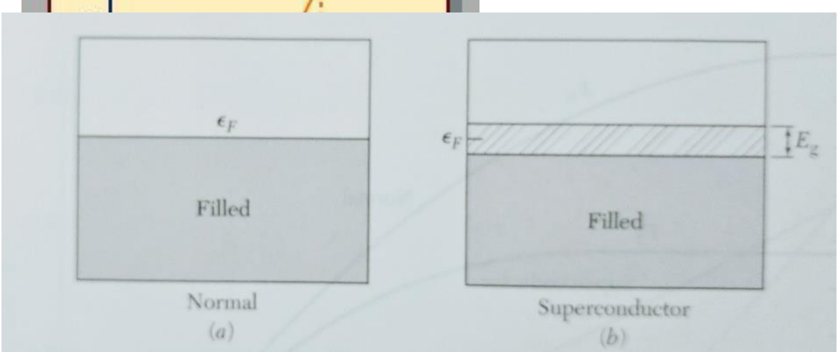
The entropy as a function of temperature is plotted to the left. There is a discontinuity at Tc, which means that there is a step function in its derivative, which is measured by heat capacity. The temperature dependence of heat capacity *below* Tc can reveal information about another property of superconductors, the superconducting energy gap.

Energy gap

Superconductors are characterized by an energy gap which is tied to the Fermi surface. This gap is different than in semiconductors because it arises from superconductivity, not from electron-lattice interactions, and it appears all around the Fermi surface, not just at the Brillouin zone boundaries.



The energy gap in a superconductor is centered around E_F by definition. In some superconductors, the gap is isotropic around the entire Fermi surface or fermi surfaces, but in others, it may be different in different directions (even zero at some points or lines on the Fermi surface called nodes) or on different Fermi surfaces.



In a superconductor, the charge carrier unit is not a single electron, but a pair of electrons called a Cooper pair, and the energy gap can physically be thought of as the energy required to break a Cooper pair.

The presence of this energy gap has consequences for other physical observables. For example, if the superconducting gap is non-zero everywhere on the Fermi surface, the superconductor is transparent to photons with energy smaller than the gap. Moreover, the energy gap (both magnitude and presence/absence of nodes) manifests in details of the temperature dependence of heat capacity below T_c.

Properties of superconductors we understand

Isotope effect

In simple metallic superconductors, for which there is a microscopic theory (later lecture), they are characterized by the fact that the superconducting transition temperature varies depending on the atomic mass (isotope) of the specimen. The variation is as follows:

$$M^{\alpha}T_c = const$$

Where M is the average atomic mass, and α is the isotope effect coefficient (a materials dependent quantity) typically ~0.5 but a bit smaller.

London equation

The London equations date to the 1930s and describe how electric and magnetic fields behave in superconductors. This section is applicable to both conventional and unconventional suprconductors.

For a superconductor with full expulsion of the magnetic field, the magnetic susceptibility is $\chi = \frac{M}{B_a} = -\frac{1}{4\pi}$ (in CGS units) or $\chi = -1$ (in SI). How do we set up electromagnetic fields inside the superconductor to make this happen? The London equation describes this, and it is applicable to all superconductors.

One way to arrive at the London equation is to postulate that in the superconducting state, the current density, **j**, is proportional to the vector potential of the magnetic field, **A** ($\boldsymbol{B} = \nabla \times \boldsymbol{A}$):

In SI units, this proportionality is written as:

$$\boldsymbol{j} = -\frac{1}{\mu_0 \lambda_L^2}\boldsymbol{A}$$

Where μ_0 is the permeability of free space, and λ_L is a constant whose physical significance will become clear in a little bit. The equation above is the London equation, and we can express it in another way by taking the curl of both sides

$$\nabla \times \boldsymbol{j} = -\frac{1}{\mu_0 \lambda_L^2}\boldsymbol{B}$$

The London equation is assumed to be written in the London gauge where $\nabla \cdot \boldsymbol{A} = 0$ and $\boldsymbol{A}_n = 0$ on any external surface through which no external current is fed. This also implies that $\nabla \cdot \boldsymbol{j} = 0$ and $\boldsymbol{j}_n = 0$

Using the Maxwell equation $\nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0 \boldsymbol{j}$ and assuming there is no time varying electric field, we get

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$

We can take the curl of both sides to obtain

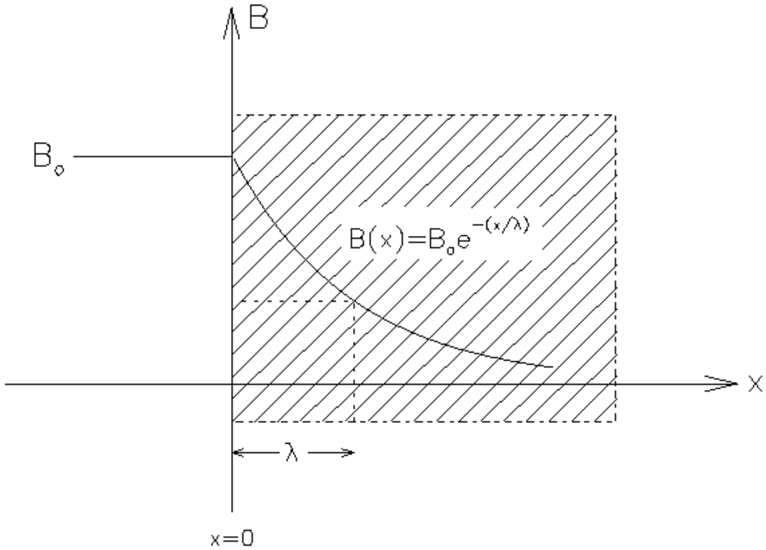
$$\nabla \times \nabla \times \boldsymbol{B} = -\nabla^2 \boldsymbol{B} = \mu_0 \nabla \times \boldsymbol{j}$$

(the second step uses the vector calculus identity $\nabla \times \nabla \times \boldsymbol{B} = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}$ where the first term on the right is zero because there are no magnetic monopoles)

This can be combined with the London equation to give:

$$\nabla^2 \boldsymbol{B} = \frac{\boldsymbol{B}}{\lambda_L^2}$$

Notably, a uniform solution ($B(r) = B_0 = const$) is not allowed unless $B_0 = 0$, and this perfectly captures the fact that a superconductor will not support and internal magnetic field. The equation from a few steps ago ensures that $j = 0$ in a region where B=0.



The solution to the equation above is a decaying exponential. Assuming a semi-infinite superconductor with magnetic field B(0) on the surface, the field inside the superconductor is given by:

$$B(x) = B(0)e^{-x/\lambda_L}$$

Thus, λ_L measures the depth of penetration of the magnetic field and it is known as the London penetration depth. Quantitatively, this materials-dependent length scale varies from 15-300 nm at zero temperature

We can also find a similar equation for current density:

$$\nabla \times \nabla^2 \boldsymbol{B} = \frac{\nabla \times \boldsymbol{B}}{\lambda_L^2}$$

$$\nabla^2(\nabla \times \boldsymbol{B}) = \frac{\nabla \times \boldsymbol{B}}{\lambda_L^2}$$

$$\nabla^2(\mu_0 \boldsymbol{j}) = \frac{\mu_0 \boldsymbol{j}}{\lambda_L^2}$$

$$\nabla^2 \boldsymbol{j} = \frac{\boldsymbol{j}}{\lambda_L^2}$$

Thus, current also decays exponentially from the surface of a superconductor, with a length set by the London penetration depth. This means that:

- If a superconductor is placed in a magnetic field, ‘screening currents’ will spontaneously occur in the superconductor to oppose this magnetic field, and these screening currents are confined only to the surface of a superconductor over a depth set by λ_L
- If a current deliberately flows through a superconductor, it will also be confined only to the surface regions

The London penetration depth is related to material parameters in the following way:

$$\lambda_L = \left(\frac{\epsilon_0 mc^2}{n_s e^2}\right)^{1/2} = \left(\frac{m}{\mu_0 n_s e^2}\right)^{1/2}$$

Where m is the effective electron mass, μ_0 and ϵ_0 are the permeability and permittivity of free space ($c^2 = 1/\mu_0\epsilon_0$), and n_s is the superfluid density—the density of electrons (N/V) which participate in superconductivity. Note that a tiny fraction of valence electrons in a metal are involved in superconductivity.

Ginzburg-Landau model

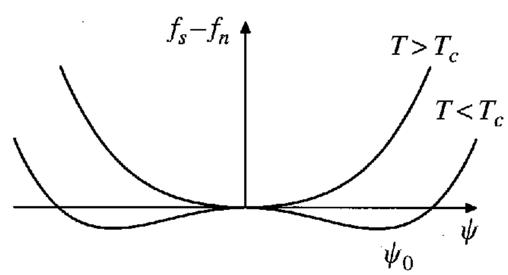
The Ginzburg-Landau model dates to the early 1950s, and it is applicable to both conventional and unconventional superconductors. The brief overview below skips some steps, and these missing steps can be found in J. Annett's book listed on p1.

The G-L theory posits an order parameter, that is zero above Tc in the normal state, and non-zero below Tc in the superconducting state. It is complex valued, which we will use in the discussion of the josephson effect at the end of these notes.

$$\psi = \begin{cases} 0, & T > T_c \\ \psi(T) \neq 0, & T < T_c \end{cases}$$

The free energy density is posited to be related to this order parameter in the following way. Note that this formalism is common to other second order phase transitions (e.g. magnetism)

$$f_s(T) - f_n(T) = a(T)|\psi|^2 + \frac{1}{2}b(T)|\psi|^4 + \cdots$$



Here, f_s is the free energy in the superconducting state, f_n is the free energy in the normal state, and both sides of the equation reflect the free energy difference between the normal and superconducting state.

The parameters a,b are phenomenological constants. To have a minimum of free energy somewhere, **b must be positive**. The sign of a will determine if there is one minimum at $\psi = 0$ (a=positive, which must be above Tc, as per the definition of order parameter) or if there are two local minima at $\psi = \pm\psi_0$ (**a=negative**, which must be below Tc). ψ_0 (location of local minima for negative a) can be found as a function of a and b by taking the derivative of the quartic equation above wrt $|\psi|$

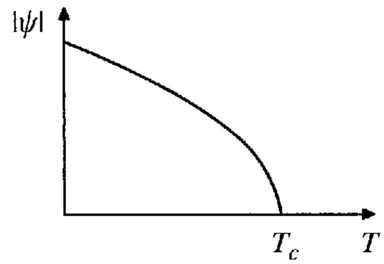
$$0 = 2a(t)|\psi| + 4 \cdot \frac{1}{2}b(T)|\psi|^3 \rightarrow |\psi|^2 = -a(T)/b(T)$$

Note that we will usually be talking about the superconducting state where a is negative.

To lowest order, a(T) and b(T) can be approximated as follows:

$$a(T) \approx \dot{a} \times (T - T_c) + \cdots$$
$$b(T) \approx b + \cdots$$

Here, $\dot{a}, b$ are constants (not temperature dependent). There is no 0th order term in the expression for a(T) to allow it to change signs across Tc. We can combine previous equations to write an expression for the temperature dependence of the order parameter near Tc:



$$\psi = \begin{cases} 0, & T > T_c \\ \left(\frac{\dot{a}}{b}\right)^{\frac{1}{2}} (T_c - T)^{1/2}, & T < T_c \end{cases}$$

Returning back to the original quartic equation, We can use earlier values of ψ_0 to solve for the value of $f_s(T) - f_n(T)$ at ψ_0….how much does the energy decrease by becoming a superconductor? Since nature has the imperative to minimize energy, the negative value will indicate that superconductivity is a thermodynamically favorable state:

$$(f_s - f_n)|_{\psi=\psi_0} = -\frac{a^2(T)}{2b(T)} = -\frac{\dot{a}^2(T - T_c)^2}{2b} = -\frac{\mu_0 H_c^2}{2}$$

The RHS of this equation is not justified in these notes (See Annett, Ch4), but $\mu_0 H_c^2/2$ expresses the condensation energy as a function of the **thermodynamic** critical field—another measure of how much energy is 'saved' by becoming a superconductor.

G-L theory allows for the order parameter to be spatially inhomogeneous, which is important at surfaces and in vortices where superconductivity is disturbed. These notes focus on the consequences of this fact.

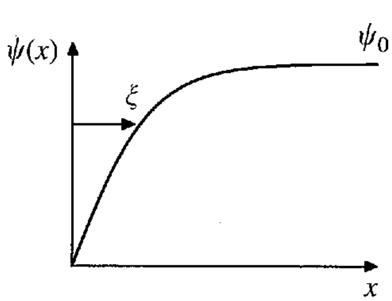
Rewrite the free energy expression to include a gradient of ψ and explicit spatial dependence:

$$f_s(T) - f_n(T) = \frac{\hbar^2}{2m^*}|\nabla\psi(\boldsymbol{r})|^2 + a(T)|\psi(\boldsymbol{r})|^2 + \frac{1}{2}b(T)|\psi(\boldsymbol{r})|^4 + \cdots$$

Here, m* is an effective mass, and there are no magnetic fields.

Now, many steps are skipped, but generally, we minimize the total free energy of the system (spatial integral of equation above) to get a Shrodinger-like equation for $\psi(r)$

$$-\frac{\hbar^2}{2m^*}\nabla^2\psi(r) + (a + b|\psi(r)|^2)\psi(r) = 0$$



Consider this equation in 1D, and make a boundary between a superconductor and a normal metal. In the metal, $\psi = 0$. Put the boundary at x=0, and x<0 is the metal side.

$$-\frac{\hbar^2}{2m^*}\frac{d^2\psi(x)}{dx^2} + a(T)\psi(x) + b(T)\psi^3(x) = 0$$

This can be solved to yield:

$$\psi(x) = \psi_0 \tanh\left(\frac{x}{\xi(T)\sqrt{2}}\right)$$

Where $\xi(T)$ is the Ginzburg-Landau coherence length. This is a 'healing' length of a superconductor…if it is disturbed locally, it will recover over this length scale, which tends to be several 10s to several 100s of nm at zero temperature.

In the solution to the shrodinger like equation above,

$$\xi(T) = \left(\frac{\hbar^2}{2m^*|a(T)|}\right)^{1/2} = \xi(0)\left|\frac{T - T_c}{T_c}\right|^{-1/2}$$

Where $\xi(0)$ is the coherence length at zero temperature. Coherence length diverges at T_c, reflecting the fact that at Tc, the superconductor will have 'infinite' healing length…you locally disturb superconductivity, and it will not recover.

Other results of GL theory (these are given without derivation, so you can know where some common expressions come from:

Superfluid density: $n_s = 2|\psi^2| = 2\,\frac{\dot{a}(T_c - T)}{b}$

This indicates that the superfluid density in the London equations is pretty much the order parameter.

London penetration depth: $\lambda(T) = \left(\frac{m_e b}{2\mu_0 e^2 \dot{a}(T_c - T)}\right)^{1/2}$

This gives the temperature dependence of the London penetration depth, which diverges at Tc

The ratio $\frac{\boldsymbol{\lambda(T)}}{\boldsymbol{\xi(T)}} = \boldsymbol{\kappa}$ determines whether a superconductor is type I (full expulsion of magnetic field) or type II (partial expulsion of magnetic field with magnetic field entering superconductor via vortices:

Type I: $\kappa < 1/\sqrt{2}$ (λ is larger)

Type II: $\kappa > 1/\sqrt{2}$ (ξ is larger)

Connection between upper critical magnetic field and coherence length:

$$\mu_0 H_{c2}(T) = \frac{\Phi_0}{2\pi\xi(T)^2}$$

Where Φ_0 is the magnetic flux quantum in a superconductor, $\Phi_0 = \frac{h}{2e}$ (more on that later). Note that H_{c2} is the magnetic field required to destroy superconductivity (implicitly via too many vortices).

This equation implies that there is exactly one flux quantum per unit area $2\pi\xi(T)^2$.

This can also be connected to H_c, the thermodynamic critical field discussed earlier in this section. Details are omitted, but it is just algebra involving equations here

$H_{c2} = \sqrt{2}\,\kappa\, H_c$

A different derivation of coherence length

The London penetration depth (λ_L) is one fundamental length in a superconductor, and the coherence length (ξ) is another. The coherence length can have several (not unrelated) physical interpretations

- In a superconductor, the charge carrier unit is two electrons called a 'Cooper pair', and these two electrons are not necessarily adjacent to each other in real space. The coherence length can be thought of as describing the physical 'size' of a Cooper pair
- If superconductivity is disturbed or destroyed locally, the coherence length represent a 'healing length' over which it will recover.
- If a superconductor is interfaced with a non-superconductor, superconductivity will be suppressed slightly over a coherence length of the interface (and superconductivity will also penetrate into the non-superconductor—the proximity effect)

A spatial variation in the state of an electronic system requires kinetic energy, as we will see in a moment. We compare a plane wave with a strongly modulated wavefunction. This derivation relates most closely to the second interpretation of the coherence length.

Plane wave: $\psi = e^{ikx}$

Modulated function: $\phi(x) = 2^{-\frac{1}{2}}\left(e^{i(k+q)x} + e^{ikx}\right)$

Probability density of plane wave is uniform in space: $\psi^*\psi = e^{-ikx}e^{ikx} = 1$

Probability density of other wavefunction is modulated with wavevector q: $\phi^*\phi = \frac{1}{2}\left(e^{-i(k+q)x} + e^{-ikx}\right)\left(e^{i(k+q)x} + e^{ikx}\right) = \frac{1}{2}\left(2 + e^{iqx} + e^{-iqx}\right) = 1 + \cos qx$

The kinetic energy of the plane wave is $\epsilon = \frac{\hbar^2 k^2}{2m}$

The kinetic energy of the modulated distribution is given by:

$$\int dx\, \phi^* \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\phi = \frac{1}{2}\frac{\hbar^2}{2m}\left[(k+q)^2 + k^2\right] \approx \frac{\hbar^2 k^2}{2m} + \frac{\hbar^2}{2m}kq$$

Assuming $q \ll k$

The increase in energy in the modulated wavefunction is $\frac{\hbar^2}{2m}kq$, and if this exceeds the superconducting gap, E_g, superconductivity is destroyed. Note that most of the superconductivity literature uses the symbol Δ to denote the superconducting gap, but your textbook uses E_g. We can solve for a critical value of q for this to happen, assuming the relevant k is k_F, because superconductivity is an instability of the Fermi surface.

$$\frac{\hbar^2}{2m}k_F q_0 = E_g$$

The intrinsic coherence length, $\xi_0 = 1/q_0$, and we can solve for it:

$$\xi_0 = \frac{\hbar^2 k_F}{2mE_g} = \frac{\hbar v_f}{2E_g}$$

Superconductivity (at least the kind that we understand) is a macroscopic quantum phenomenon that is quite robust against impurities (at least non-magnetic ones), but impurities do decrease the effective coherence length. If the mean free path (measured in the normal state) is given by ℓ and is smaller than ξ_0, the two length scales in superconductors are written in the following way:

Coherence length: $\xi = (\xi_0 \ell)^{1/2}$

Magnetic penetration depth: $\lambda = \lambda_L \left(\frac{\xi_0}{\ell}\right)^{1/2}$

BCS theory of superconductivity

BCS (Bardeen-Cooper-Schrieffer) theory explains superconductivity in metals and intermetallic compounds, and this lecture will provide an overview of this theory. A few results to look forward to:

- Origin of the energy gap, how this relates to Cooper pairs, and equation to calculate its magnitude
- Role of Fermi surface and electron-phonon coupling in superconductivity
- $T_c = 1.14\,\Theta e^{-1/UD(\epsilon_F)}$ where Θ is the debye temperature (involvement of phonons), $D(\epsilon_F)$ is the density of states at the Fermi energy (only electrons close to the Fermi energy matter), and U is an attractive electron-phonon interaction.

As a historical note, superconductivity was discovered in 1911, but BCS theory was not published until 1957. In the interim, there were many intriguing wrong theories (see: https://arxiv.org/abs/1008.0447). In the years preceding BCS theory, there were a number of smoking gun experiments that pointed scientists in the correct direction:

- Observation of the isotope effect (https://journals.aps.org/pr/abstract/10.1103/PhysRev.78.477); read this paper just for the opening paragraphs
- Exponential behavior of low-temperature heat capacity, which implied an energy gap for the lowest energy excitations (https://journals.aps.org/pr/abstract/10.1103/PhysRev.96.1442.2)

The first key idea in BCS theory is that there is an effective attraction for electrons near the Fermi energy. Normally, electrons repel each other:

$$V(\boldsymbol{r} - \boldsymbol{r}') = \frac{e^2}{4\pi\epsilon_0|\boldsymbol{r} - \boldsymbol{r}'|}$$

While this is true for free electrons, in a metal, this interaction is screened, and the Thomas-Fermi model is the simplest model to describe this, giving an effective interaction of the form:

$$V_{TF}(\boldsymbol{r} - \boldsymbol{r}') = \frac{e^2}{4\pi\epsilon_0|\boldsymbol{r} - \boldsymbol{r}'|} e^{-|\boldsymbol{r}-\boldsymbol{r}'|/r_{TF}}$$

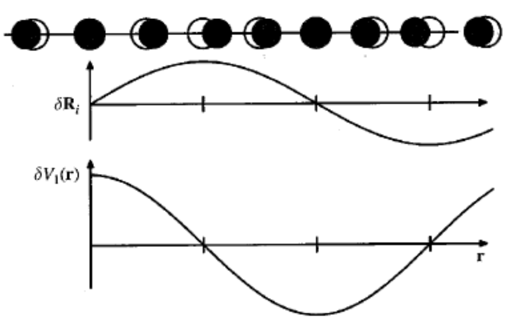
Where r_{TF} is the Thomas-fermi screening length. This gives a repulsive force that is much shorter range.

The second ingredient is electrons interacting with the lattice.

As we saw in chapter 5, phonons in a crystal lattice can be treated as a set of quantum harmonic oscillators. The Hamiltonian for this can be written as:

$$H=\sum_{\boldsymbol{q},\lambda}\hbar\omega_{\boldsymbol{q}\lambda}\left(a_{\boldsymbol{q}\lambda}^{+}a_{\boldsymbol{q}\lambda}+\frac{1}{2}\right)=\sum_{\boldsymbol{q},\lambda}\hbar\omega_{\boldsymbol{q}\lambda}\left(n_{q\lambda}+\frac{1}{2}\right)=$$

Where **q** is the wavevector of the phonon and λ labels the phonon mode (branch), in the case that there is more than one phonon with the same **q**. In a 3D solid with N atoms per unit cell, there are 3N phonon branches in total. The harmonic oscillator ladder operators can be used to calculate atomic displacement for an atom located at position $\boldsymbol{R}_i$ due to phonons:



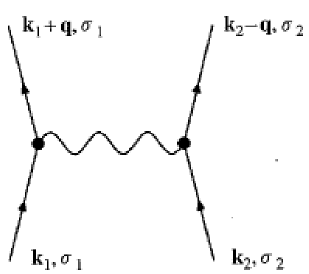
$$\delta\boldsymbol{R}_i=\sum_{\boldsymbol{q}\lambda}\boldsymbol{e}_{q\lambda}\left(\frac{\hbar}{2M\omega_{\boldsymbol{q}\lambda}}\right)^{\frac{1}{2}}(a_{\boldsymbol{q}\lambda}^{+}+a_{\boldsymbol{q}\lambda})e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}$$

Where M is the atomic mass (assuming only one atom per unit cell), and $\boldsymbol{e}_{q\lambda}$ is the unit vector of the **atomic displacement** of phonon mode $\boldsymbol{q}\lambda$.

This modulation in the atomic position will also modulate the atomic potential of the lattice:

$$\delta V_1(\boldsymbol{r})=\sum_i\frac{\partial V_1(\boldsymbol{r})}{\partial\boldsymbol{R}_i}\delta\boldsymbol{R}_i$$

This is a periodic modulation of the lattice with wavelength $2\pi/q$, and an electron moving through this potential will experience diffraction. If it is initially in Bloch state $\psi_{\boldsymbol{k}}(\boldsymbol{r})$ it will be diffracted to Bloch state $\psi_{\boldsymbol{k}-\boldsymbol{q}}(\boldsymbol{r})$. The extra momentum has been provided by the phonon. One can either think of this process as creating a phonon with momentum **q** or annihilating one with momentum $-\boldsymbol{q}$. A second electron can also interact with this phonon, and by reciprocity, these electrons then interact with each other through exchange of a photon.



The effective interaction between electrons due to exchange of a phonon is given by:

$$V_{eff}(\boldsymbol{q},\omega)=\left|g_{\boldsymbol{q}\lambda}\right|^2\frac{1}{\omega^2-\omega_{\boldsymbol{q}\lambda}^2}$$

Where $g_{\boldsymbol{q}\lambda}$ is the matrix element for scattering an electron from state **k** to state **k+q**. It quantifies the strength of electron-phonon coupling for phonon $\boldsymbol{q}\lambda$, or the probability for an electron to be scattered by phonon $\boldsymbol{q}\lambda$. It turns out that $g_{\boldsymbol{q}\lambda}$ is of order $\sqrt{m/M}$ where m is the electron mass and M is the atomic mass. Because electrons are much lighter than atoms, we can think of electrons and phonons being weakly coupled, which allows for this simplified derivation to be valid.

Further simplifications can be made to the equation above to achieve an approximate solution that captures the key physics:

- Replace $g_{\boldsymbol{q}\lambda}$ by g_{eff}, a q-independent average value for electron-phonon coupling

- Replace $\omega_{\boldsymbol{q}\lambda}$ by a 'typical' phonon frequency, which is usually taken to be the debye frequency ω_D

This gives an effective electron-phonon interaction as:

$$V_{eff} = \left|g_{eff}\right|^2 \frac{1}{\omega^2 - \omega_D^2}$$

Notice that this interacting is attractive (negative) for frequencies less than ω_D. Because superconductivity is a low temperature phenomenon, typically only low frequencies ($\ll \omega_D$) will be relevant. Thus, the interaction is always attractive, and can be rewritten in its final form, $V_{eff} = -\left|g_{eff}\right|^2$. The key insight of BCS theory is that electrons near the Fermi surface are susceptible to attractive interactions with one another.

The corresponding term in the Hamiltonian, which we will use in the next section, is given by:

$$H_1 = -\left|g_{eff}\right|^2 \sum c^{+}_{\boldsymbol{k_1}+\boldsymbol{q}\sigma_1} c^{+}_{\boldsymbol{k_2}-\boldsymbol{q}\sigma_2} c_{\boldsymbol{k_1}\sigma_1} c_{\boldsymbol{k_2}\sigma_2}$$

Which corresponds to scattering an electron from momentum **k1** and spin σ_1 to momentum **k1+q** and an electron from momentum **k2** and spin σ_2 to momentum **k2-q**

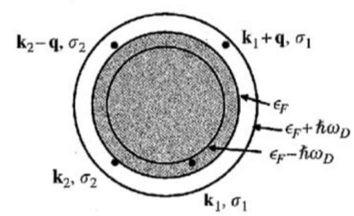


Fig. 6.6 The effective electron–electron interaction near the Fermi surface. The electrons at $\mathbf{k}_1, \sigma_1$ and $\mathbf{k}_2, \sigma_2$ are scattered to $\mathbf{k}_1 + \mathbf{q}, \sigma_1$ and $\mathbf{k}_2 - \mathbf{q}, \sigma_2$. The interaction is attractive provided that all of the wave vectors lie in the range where $\epsilon_{\mathbf{k}}$ is within energy $\pm\hbar\omega_D$ of the Fermi energy.

Cooper pairs

The next step after the presence of an attractive interaction is the formation of pairs. The charge carrier unit in a superconductor is not a single electron, but a pair of electrons called a Cooper pair.

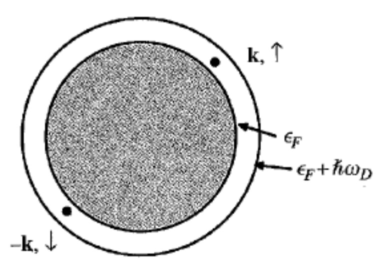


Fig. 6.7 The Cooper problem: two electrons outside a fully occupied Fermi sea. The interaction is attractive provided that the electron energies are in the range $\epsilon_F < \epsilon_{\mathbf{k}} < \epsilon_F + \hbar\omega_D$.

Consider a spherical Fermi surface where all states $k < k_F$ are occupied. Now consider placing two extra electrons outside the Fermi surface. The two-particle wavefunction of these extra electrons is:

$$\Psi(\boldsymbol{r_1}, \sigma_1, \boldsymbol{r_2}, \sigma_2) = e^{ik_{cm}\cdot R_{cm}} \varphi(\boldsymbol{r_1} - \boldsymbol{r_2}) \phi^{spin}_{\sigma_1,\sigma_2}$$

Where R_{cm} is the center of mass position of the pair, $\hbar k_{cm}$ is the total momentum of the pair. φ is the spatial part of the two-electron wavefunction and ϕ is the spin part. The spin part is usually a singlet:

$$\phi^{spin}_{\sigma_1,\sigma_2} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow> - |\downarrow\uparrow>)$$

Fermion antisymmetry implies that $\Psi(\boldsymbol{r_1}, \sigma_1, \boldsymbol{r_2}, \sigma_2) = -\Psi(\boldsymbol{r_2}, \sigma_2, \boldsymbol{r_1}, \sigma_1)$. Since the spin function is odd with respect to exchange of particles, the spatial part must be even.

Expanding $\varphi(\boldsymbol{r_1} - \boldsymbol{r_2})$ in terms of bloch waves gives:

$$\varphi(\boldsymbol{r_1}-\boldsymbol{r_2}) = \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot(r_1-r_2)}$$

Where φ_k are some expansion coefficients to be found, given the constraint $C = \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}}$

The full pair wave function can be written as a sum of slater determinants

$$\Psi(\boldsymbol{r_1},\sigma_1,\boldsymbol{r_2},\sigma_2) = \sum_k \varphi_{\boldsymbol{k}} \begin{vmatrix} \psi_{\boldsymbol{k}\uparrow}(\boldsymbol{r_1}) & \psi_{\boldsymbol{k}\downarrow}(\boldsymbol{r_2}) \\ \psi_{-\boldsymbol{k}\uparrow}(\boldsymbol{r_1}) & \psi_{-\boldsymbol{k}\downarrow}(\boldsymbol{r_2}) \end{vmatrix}$$

Where terms like $\psi_{\boldsymbol{k}\downarrow}(\boldsymbol{r_2})$ represent single particle bloch states. The form of the slater determinant implies that Cooper pairs involve electrons at **k** and **-k** which have opposite spins.

Plugging in this trial wavefunction into the schrodinger equation, including both a kinetic energy term and the interaction term from earlier yields:

$$E\varphi_{\boldsymbol{k}} = 2\epsilon_{\boldsymbol{k}}\varphi_{\boldsymbol{k}} - \left|g_{eff}\right|^2 \sum_{\boldsymbol{k'}} \varphi_{\boldsymbol{k'}}$$

Where $\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$

Using $C = \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}}$ we can solve the equation above to yield

$$\varphi_{\boldsymbol{k}} = -C\left|g_{eff}\right|^2 \frac{1}{E-2\epsilon_k}$$

Self consistency requires that

$$C = \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}} = -C\left|g_{eff}\right|^2 \sum_k \frac{1}{E-2\epsilon_k}$$

Or equivalently:

$$1 = -\left|g_{eff}\right|^2 \sum_k \frac{1}{E-2\epsilon_k}$$

We can convert the sum into an integral over energy via the density of states, and we take this integral only up to $\hbar\omega_D$, because interactions cease to be attractive beyond this point. The density of states is taken outside the integral because it is assumed that it does not change much over the narrow energy range being considered.

$$1 = -\left|g_{eff}\right|^2 D(\epsilon_F) \int_0^{\hbar\omega_D} d\epsilon \frac{1}{E-2\epsilon}$$

The integral can be solved and the results can be arranged to give

$$-E = 2\hbar\omega_D e^{-1/\left|g_{eff}\right|^2 D(\epsilon_F)}$$

This equation implies that a **bound state** exists, meaning that it is more energetically favorable for electrons to be bound in a pair than to be separated. The energy scale is set by both the debye frequency, and the density of states at ϵ_F multiplied by the electron-phonon coupling parameter. Note that your textbook uses U to refer to $|g_{eff}|^2$.

Superconducting gap

The derivation of the BCS gap parameter, Δ, is very much abridged, but details can be found in supplementary reading posted on Canvas (Annett).

The BCS gap parameter can be expressed as the expectation value of the Cooper pair operator, which corresponds to removing electrons of opposite spin from opposite sides of the fermi surface to join the superconducting condensate:

$$\Delta = |g_{eff}|^2 \sum_{\boldsymbol{k}} < c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} > = |g_{eff}| \sum_{\boldsymbol{k}} \frac{\Delta}{2E_{\boldsymbol{k}}}$$

Where $E_k = \sqrt{(\epsilon_k - \mu)^2 + |\Delta|^2}$. Physically, $\pm E_k$ represent band dispersions in the superconducting state. The expression above is given without derivation, but the derivation can be found in the resource mentioned earlier. Δ can be cancelled from the first and third term above to yield:

$$1 = \frac{|g_{eff}|^2}{2} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{(\epsilon_k - \mu)^2 + |\Delta|^2}}$$

This expression can be used to find the magnitude of the BCS gap parameter, Δ. As before, we transform the sum over k into an integral over energy using the density of states, and we assume that our energy range is small enough such that the density of states can be approximated as a constant, $D(\epsilon_F)$. We have also set the chemical potential μ to be zero.

$$1 = |g_{eff}|^2 D(\epsilon_F) \int_0^{\hbar\omega_D} \frac{1}{\sqrt{\epsilon^2 + |\Delta|^2}} d\epsilon$$

This integral can be solved approximately to yield:

$$|\Delta| = 2\hbar\omega_D e^{-1/|g_{eff}|^2 D(\epsilon_F)}$$

This is similar to an expression in your textbook for T_c, and you will connect the two in your homework. Physically, the BCS gap parameter |Δ| is related to the order parameter of the superconducting state. It is zero above Tc, and increases below Tc until it saturates at low temperature. Its temperature dependence is captured by the following equation, also given without derivation:

$$1 = |g_{eff}|^2 D(\epsilon_F) \int_0^{\hbar\omega_D} \frac{1}{\sqrt{\epsilon^2 + |\Delta|^2}} \tanh \frac{\sqrt{\epsilon^2 + |\Delta|^2}}{2k_B T} d\epsilon$$

By taking the limit where $\Delta \rightarrow 0$, one can obtain the expression for Tc in Kittell:

$$k_B T_c = 1.13\hbar\omega_D e^{-1/|g_{eff}|^2 D(\epsilon_F)}$$

This has very similar form to the equation for the gap because both describe the robustness of the superconducting state

At T=0, one can also derive the relationship between $\Delta(T=0)$ and Tc:

$$2\Delta(T=0) = 3.52 k_B T_c$$

Note that all the expressions with very specific numerical prefactors are only applicable in a 'weak coupling' regime, but they are nevertheless a useful benchmark.

Phase of superconducting order parameter and Josephson effect

The superconducting order parameter is a complex order parameter that has both amplitude and phase.

The wavefunction is also complex and is written in terms of n_s (the density of *superconducting electrons*)

$$\psi = \sqrt{n_s}\, e^{i\theta(\boldsymbol{r})}$$

The prefactor is materials-dependent and temperature-dependent. It is equal to zero above Tc and reaches its maximum value at T=0. Earlier in the notes, n_s was defined in such a way that $\psi = \sqrt{n_s/2}$, but it is redefined here to incorporate that factor of 2 into the superfluid density; this is for ease of notation in upcoming derivations. The exponent term is spontaneously chosen when a material cools down into the superconducting state, similar to how a ferromagnet spontaneously chooses a magnetization direction. As it is written above, the phase may have a position-dependence, which is related to current flow, but it does not always. It can have a momentum dependence around the Fermi surface, $\theta(k)$ (more on that later).

The phase has the following physical consequences

The velocity of a quantum mechanical charged particle can be expressed as follows:

$$\boldsymbol{v} = \frac{1}{m}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right) = \frac{1}{m}(-i\hbar\nabla - \frac{q}{c}\boldsymbol{A})$$

The particle flux is given by:

$$\psi^* \boldsymbol{v} \psi = \frac{n_s}{m}(\hbar\nabla\theta - \frac{q}{c}\boldsymbol{A})$$

And the current density is given by:

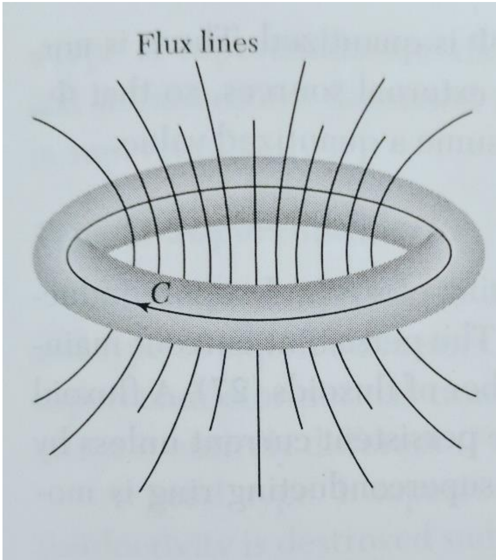
$$\boldsymbol{j} = \psi^* \boldsymbol{v} \psi = \frac{n_s q}{m}(\hbar\nabla\theta - \frac{q}{c}\boldsymbol{A})$$

Take the curl of both sides, and note that the curl of the gradient of a scalar is zero.

$$\nabla \times \boldsymbol{j} = -\frac{n_s q^2}{mc}\boldsymbol{B}$$

This is one form of the London equation.

Consider now a superconducting ring. Take a closed path C through the **interior** of the superconducting material well away from the surface. As we learned earlier, current and magnetic field are confined only



to the surface region of a superconductor, so **B** and **j** are zero along our contour. Plugging this in to the earlier equation for **j** yields:

$$\hbar c \nabla\theta = \boldsymbol{q}A$$

The change in phase as one traverses around the ring is given by:

$$\oint_C \nabla\theta \cdot dl = \theta_2 - \theta_1$$

Lets say that our contour represents one revolution around the ring. We must come back to the same value of ψ after we traverse the ring. This means that

$$\theta_2 - \theta_1 = 2\pi s$$

Where s is an integer. We can now take the contour integral on the other side and use Stokes theorem (as we did in Ch9) to change an integral around a loop to a surface integral.

$$\oint_C \boldsymbol{A} \cdot dl = \int_C (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{\sigma} = \int \boldsymbol{B} \cdot d\boldsymbol{\sigma} = \Phi$$

Where $d\sigma$ is an element of area on the surface bounded by C, and Φ is the magnetic flux through C. Equating the two contour integrals, we get:

$$\hbar c 2\pi s = q\Phi$$

$$\Phi = 2\pi\hbar c s/q$$

The charge carrier unit in a superconductor is a Cooper pair of two electrons, so we should use q=-2e. This gives the flux inside the superconducting loop as

$$\Phi = s\Phi_0$$

Where Φ_0 is the magnetic flux quantum, given by $\frac{2\pi\hbar c}{2e} = \frac{hc}{2e}$ (cgs) or $\frac{2\pi\hbar}{2e} = \frac{h}{2e}$ (SI).

In general, the flux through a superconductor can come both from the current in the superconducting loop or an externally applied field. **No matter what, the flux through the superconducting loop is quantized**. In the presence of an external field, the current inside the superconductor will adjust to make this true

$$\Phi = \Phi_{external} + \Phi_{sc} = s\Phi_0$$

Additionally, in the absence of resistance, any current that is initiated in a superconducting loop will continue on indefinitely (at least in a type I superconductor; vortices in a type II superconductors can render this statement untrue). This is called a persistent current.

We can calculate how long this persistent current should survive. The probability per unit time that a superconducting loop will lose one quantum of flux (and hence the current will change) is given by:

P=(attempt frequency)*(activation barrier)

The activation barrier at temperature T is given by:

$$\sim e^{\Delta F/k_B T}$$

Where the free energy of the barrier is given by:

$$\Delta F = (minimum\ volume\ that\ turns\ normal) * (excess\ free\ energy\ density\ of\ normal\ state)$$

The volume of the ring that must turn normal (cease being superconducting) to allow a quantum of magnetic flux to escape is $\sim R\xi^2$ where ξ is the coherence length of the superconductor and R is the thickness of the wire. The excess free energy of the normal state is given by: $H_c^2/8\pi$. This gives an approximate value of the excess free energy:

$$\Delta F \approx R\xi^2 H_c^2/8\pi$$

Lets add some approximate values for these terms, consistent with experiment:

- Wire thickness: $10^{-4}cm$
- $\xi = 10^{-4}cm$
- $H_c = 10^3$ gauss

This gives $\Delta F \approx 10^{-7}$ ergs. To calculate the activation barrier, we need some guess for the temperature, and 0.1-10K is a reasonable range.

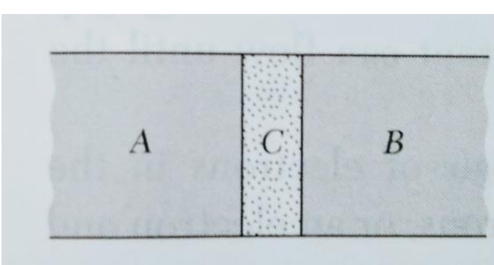
$$e^{\Delta F/k_B T} \approx e^{-10^8} \approx 10^{-4.3\times10^7}$$

The attempt frequency (in general) can be estimated from a characteristic energy scale and $\hbar$. In a superconductor, a characteristic energy scale is the superconducting gap, E_g, which is on the order of several meV or smaller. This gives an attempt frequency of $10^{12}s^{-1}$. Altogether, this gives a probability of a superconducting ring to lose one quantum of magnetic flux to be:

$$P \approx 10^{-4.3\times10^7}$$

This is much older than the age of the universe. Thus, a current that is set up in a superconducting ring will persist forever, unless external conditions such as temperature or (applied) magnetic field change.

Single particle tunneling



As we learn early in a quantum mechanics class, quantum particles can tunnel through a potential barrier that is not infinitely high, and this is true for Cooper pairs as well. Consider two metals that are separated from one another by a thin insulating barrier. If neither of these metals are superconductors, the IV characteristic of the device will be ohmic: V=IR. However, if **one** of these metals becomes a superconductor, the IV characteristic will be very different.

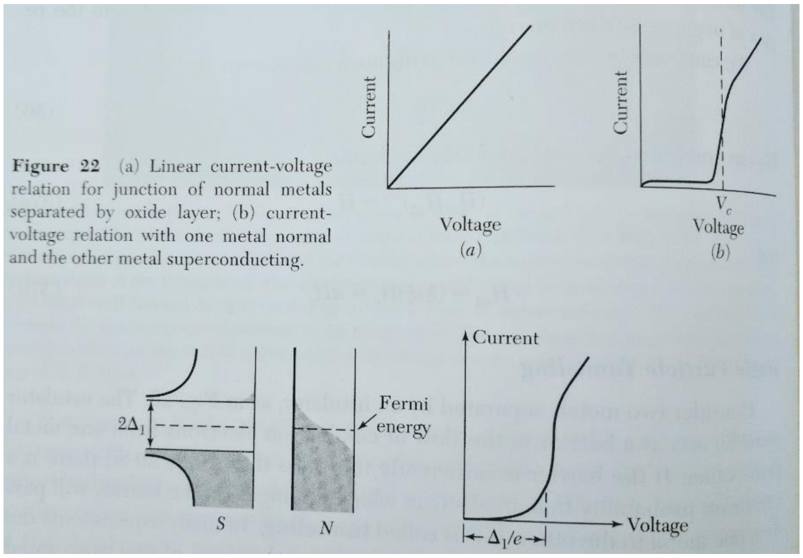


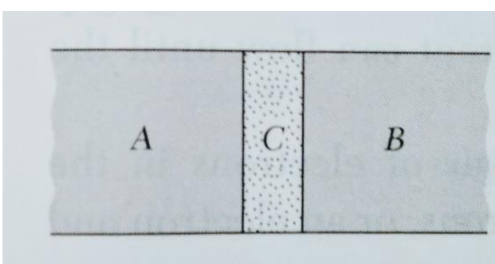
Figure 22 (a) Linear current-voltage relation for junction of normal metals separated by oxide layer; (b) current-voltage relation with one metal normal and the other metal superconducting.

There will be very little current flow for a voltage up until a critical voltage V_c. After that, the junction will look ohmic again. The voltage where current starts to flow correspond to the energy required to break a Cooper pair: $V = \frac{E_g}{2e} = \frac{\Delta}{e}$. This is one way of measuring the magnitude of a superconducting gap. At zero temperature, the onset of current flow is abrupt, but at elevated temperatures (still superconducting), the transition gets rounded.

If both sides of the insulating barrier are superconductors, this is called a josephson junction. Several things can happen in this case, and three examples are given below:

- DC josephson effect: DC current flows across junction in the absence of electric or magnetic fields
- AC josephson effect: DC voltage applied across the junction causes current oscillations across the junction
- Macroscopic quantum interference: a device made out of two josephson junctions can show interference effects as a function of magnetic field

DC Josephson effect



The wavefunction of a superconductor is written in terms of n_s (the density of *superconducting electrons*)

$$\psi = \sqrt{n_s} e^{i\theta(\mathbf{r})}$$

When a superconductor is cooled below Tc, it spontaneously chooses a phase, θ. In a josephson junction, the superconductors on either side of the barrier will generically choose different phases. The DC josephson effect is a consequence of this.

Let ψ_1 be the wavefunction on one side of the josephson junction, characterized by phase θ_1 and superfluid density $n_{s,1}$ and ψ_2 be the wavefunction on the other side, characterized by phase θ_2 and superfluid density $n_{s,2}$

The time dependent Schrodinger equation applied to these two superconductors gives:

$$i\hbar \frac{\partial \psi_1}{\partial t} = \hbar T \psi_2$$

$$i\hbar \frac{\partial \psi_2}{\partial t} = \hbar T \psi_1$$

Where $\hbar T$ represents a superconducting pair tunneling across the barrier. Because of quantum tunneling, the superconducting state in superconductor 2 leaks into superconductor 1 and vis versa, as long as the barrier is thin enough. Plugging in the full expression for the wavefunction yields:

$$\frac{\partial \psi_1}{\partial t} = \frac{1}{2} n_{s,1}^{-\frac{1}{2}} e^{i\theta_1} \frac{\partial n_{s,1}}{\partial t} + i\psi_1 \frac{\partial \theta_1}{\partial t} = -iT\psi_2$$

$$\frac{\partial \psi_2}{\partial t} = \frac{1}{2} n_{s,2}^{-\frac{1}{2}} e^{i\theta_2} \frac{\partial n_{s,2}}{\partial t} + i\psi_2 \frac{\partial \theta_2}{\partial t} = -iT\psi_1$$

Multiply both equations by $n_{s,1,2}^{\frac{1}{2}} e^{-i\theta_{1,2}}$ (choose option (1) for the first equation and option (2) for the 2nd. Also, $\delta \equiv \theta_2 - \theta_1$

$$\frac{1}{2}\frac{\partial n_{s,1}}{\partial t} + in_1 \frac{\partial \theta_1}{\partial t} = -iT\left(n_{s,1} n_{s,2}\right)^{\frac{1}{2}} e^{i\delta}$$

$$\frac{1}{2}\frac{\partial n_{s,2}}{\partial t} + in_2 \frac{\partial \theta_2}{\partial t} = -iT\left(n_{s,1} n_{s,2}\right)^{\frac{1}{2}} e^{-i\delta}$$

Now equate the real and imaginary parts of both equations:

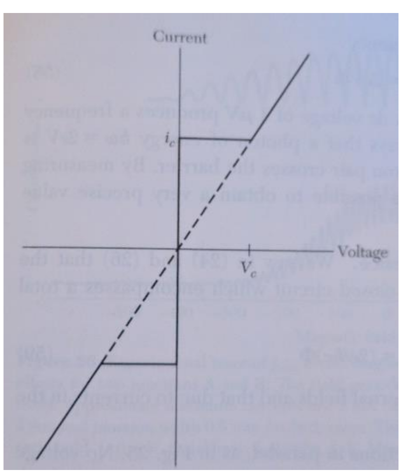
$$\frac{\partial n_{s,1}}{\partial t} = 2T\left(n_{s,1} n_{s,2}\right)^{\frac{1}{2}} \sin\delta$$

$$\frac{\partial n_{s,2}}{\partial t} = -2T\left(n_{s,1} n_{s,2}\right)^{\frac{1}{2}} \sin\delta$$

$$\frac{\partial \theta_1}{\partial t} = -T\left(\frac{n_{s,2}}{n_{s,1}}\right)^{\frac{1}{2}} \cos\delta$$

$$\frac{\partial \theta_2}{\partial t} = -T\left(\frac{n_{s,1}}{n_{s,2}}\right)^{\frac{1}{2}} \cos\delta$$

We can simplify this problem by assuming that the superconductor on both sides of the junction is the same material, such that $n_{s,1} = n_{s,2} = n_s$. This gives us the following relations:



$$\frac{\partial \theta_1}{\partial t} = \frac{\partial \theta_2}{\partial t}$$

$$\frac{\partial}{\partial t}(\theta_2 - \theta_1) = 0$$

$$\frac{\partial n_{s,2}}{\partial t} = -\frac{\partial n_{s,1}}{\partial t}$$

This is essentially a statement that superfluid density overall is conserved. The current flow from one superconductor to another is related to the change in superfluid density, which gives the following for the junction current:

$$J = J_0 \sin\delta = J_0 \sin(\theta_2 - \theta_1)$$

J_0 is proportional to the transfer interaction T. What the equation above says is that current will flow in a josephson junction in the absence of any applied voltage. This is the DC josephson effect.

The IV curve of a josephson junction is shown above. At zero bias voltage, a current i_c flows. For voltages above V_c, the junction has a finite resistance (notice the ohmic IV behavior), and this is the regime of the AC josephson effect.

AC Josephson effect

Whereas the DC Josephson effect occurs in the absence of applied voltage, the AC josephson effect requires an applied voltage. Normally, a superconductor cannot support a voltage across it (V=IR), but because a josephson junction has an insulator inside, it can (the voltage drop happens across the insulator not across the superconductor).

When crossing an insulating barrier, the Cooper pair will gain or lose energy 2eV. Lets assume that the pair is at potential -eV on one side of the barrier and +eV on the other side. The equations of motion are:

$$i\hbar\frac{\partial\psi_1}{\partial t} = \hbar T\psi_2 - eV\psi_1$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = \hbar T\psi_1 + eV\psi_2$$

Replace with values of $\psi_1 = \sqrt{n_{s,1}}e^{i\theta_1(\boldsymbol{r})}$ and $\psi_2 = \sqrt{n_{s,2}}e^{i2(\boldsymbol{r})}$:

$$\frac{1}{2}\frac{\partial n_{s,1}}{\partial t} + in_1\frac{\partial\theta_1}{\partial t} = \frac{ieVn_{s,1}}{\hbar} - iT\left(n_{s,1}n_{s,2}\right)^{\frac{1}{2}}e^{i\delta}$$

$$\frac{1}{2}\frac{\partial n_{s,2}}{\partial t} + in_2\frac{\partial\theta_2}{\partial t} = \frac{-ieVn_{s,2}}{\hbar} - iT\left(n_{s,1}n_{s,2}\right)^{\frac{1}{2}}e^{-i\delta}$$

As before, set the real and imaginary parts to be equal:

$$\frac{\partial n_{s,1}}{\partial t} = 2T\left(n_{s,1}n_{s,2}\right)^{\frac{1}{2}}\sin\delta$$

$$\frac{\partial n_{s,2}}{\partial t} = -2T\left(n_{s,1}n_{s,2}\right)^{\frac{1}{2}}\sin\delta$$

$$\frac{\partial\theta_1}{\partial t} = (\frac{eV}{\hbar}) - T\left(\frac{n_{s,2}}{n_{s,1}}\right)^{\frac{1}{2}}\cos\delta$$

$$\frac{\partial\theta_2}{\partial t} = -(\frac{eV}{\hbar}) - T\left(\frac{n_{s,1}}{n_{s,2}}\right)^{\frac{1}{2}}\cos\delta$$

If we again assume that $n_{s,1} = n_{s,2} = n_s$

$$\frac{\partial}{\partial t}(\theta_2 - \theta_1) = \frac{\partial}{\partial t}\delta = -2eV/\hbar$$

Integrate this equation to get:

$$\delta(t) = \delta(0) - 2eVt/\hbar$$

The superconducting current is thus given by a similar result as earlier, except with an evolving phase:

$$J = J_0 \sin[\delta(0) - 2eVt/\hbar]$$

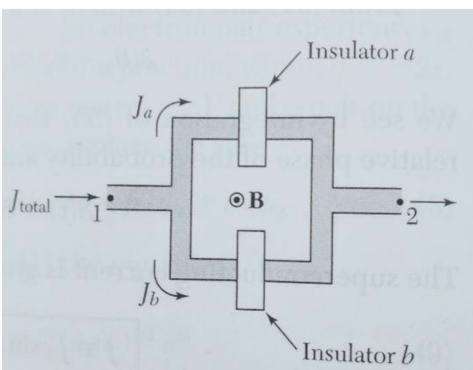
This is an oscillating (AC) current with a frequency $\omega = \frac{2eV}{\hbar}$. Another way to state this is that the measured voltage is related to the frequency of the oscillation and fundamental constants: $V = \frac{\hbar\omega}{2e}$.

The AC josephson effect has been used a standard for e/h, related to the magnetic flux quantum (h/2e), but it is more commonly used as a voltage standard, the **josephson voltage standard.** Most units have a standard yardstick related to fundamental physical processes that do not depend on the laboratory or the time. For example, the frequency (and time) standard, called the cesium standard, comes from measurements of transitions between two hyperfine ground states in Cs-133. With this frequency standard and the AC josephson effect, a josephson junction serves as the voltage standard.

Macroscopic quantum interference

This topic combines the josephson junction (two of them, actually) with earlier concepts of flux through a superconducting loop. The change in superconducting phase around a closed circuit is given by:

$$\theta_2 - \theta_1 = \left(\frac{2e}{\hbar c}\right)\Phi$$



Consider two Josephson junction in parallel as in the image on the left. This device is called a SQUID or superconducting quantum interference device.

The total current goes from point (1) to point (2), but it can take one of two paths to get there: through junction a or through junction b.

$\delta_a \equiv$ phase difference between points (1) and (2) when current goes through junction a

$\delta_b \equiv$ phase difference between points (1) and (2) when current goes through junction b

The superconducting loop can also accommodate magnetic flux Φ inside, both from an external source and from the current in the loop. This puts restrictions on the relation between δ_a and δ_b

$$\delta_b - \delta_a = \left(\frac{2e}{\hbar c}\right)\Phi$$

$$\delta_b = \delta_0 + \frac{e}{\hbar c}\Phi$$

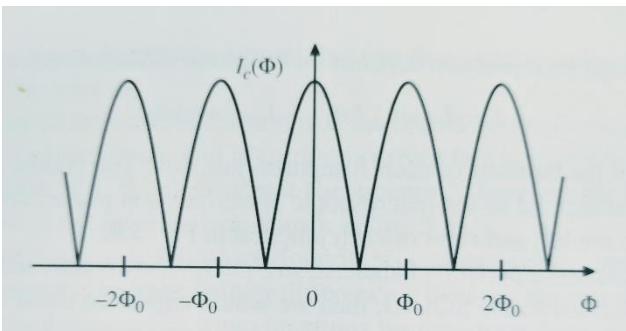
$$\delta_a = \delta_0 - \frac{e}{\hbar c}\Phi$$

The total current is the sum of the current in branch a and branch b, and each of those is given by equations for a single josephson junction discussed earlier

$$J_{total} = J_a + J_b = J_0\left\{\sin\left(\delta_0 + \frac{e}{\hbar c}\Phi\right) + \sin\left(\delta_0 - \frac{e}{\hbar c}\Phi\right)\right\} = 2J_0 \sin\delta_0 \cos\frac{e\Phi}{\hbar c}$$

Because the SQUID ring current will always have the same sign as the driving current, the expression above needs to always be positive.

$$J_{total} = 2J_0 \sin\delta_0 |\cos\frac{e\Phi}{\hbar c}|$$



The current varies with Φ and has maxima when

$$\frac{e\Phi}{\hbar c} = s\pi$$

Where s is an integer. Note that these values of Φ_{max} correspond to integer multiples of the magnetic flux quantum ($\Phi_0 = \frac{2\pi\hbar c}{2e}$). Also notice that in *this* configuration of a superconducting loop, where there are two josephson junctions breaking up the loop, the flux through the loop is *not necessarily* quantized.

The current as a function of magnetic flux appears like the image above. This looks exactly like a Fraunhoffer interference pattern in a double slit experiment. Thus, this is a demonstration of macroscopic quantum interference, where the interfering waves are the superconducting condensates in the two branches on the SQUID. Remember that the wavefunction of a superconductor looks an awful lot like the wavefunction of a plane wave.

The SQUID is one of the most applications of superconductivity, as it is used to measure magnetic flux or magnetic field. The SQUID can be made fairly large (~1cm^2), such that the flux changing by one magnetic quantum corresponds to a *tiny* change in magnetic field. One can then count the number of minima observed in the current through the SQUID to detect tiny changes in magnetic field to high accuracy.

Superconducting phase, pairing symmetries, and different types of superconductors

Most of the discussion so far has been either applicable to all superconductors or applicable to superconductors we “understand” (those explained by BCS theory). This section is primarily useful for “unconventional” superconductors, many of which have higher order pairing symmetry.

Earlier in the notes, we discussed superconducting gap (roughly the energy to break a Cooper pair), and the superconducting gap was characterized by one number. This is because conventional superconductors have an isotropic superconducting gap that is the same magnitude and the same phase (excepting current flow) everywhere. This is not necessarily true for unconventional superconductors.

In a superconductor the Cooper pairs can either be singlet (usually they are this S=0) or triplet (S=1). The antisymmetric singlet state must be accompanied by a symmetric (even) orbital component; conversely, an triplet pairing in the spin channel would correspond to an odd orbital component. Some examples:

Singlet cooper pair, s-wave ($\ell = 0$) orbital component; conventional superconductors are in this category.

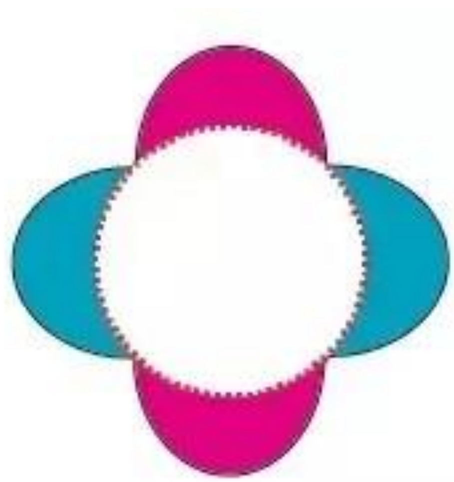
Triplet Cooper pair, p-wave ($\ell = 1$); Sr_2RuO_4 was once believed to be in this category, but the jury is out now

Singlet Cooper pair, d-wave ($\ell = 2$); cuprate high temperature superconductors are famously in this category

Triplet Cooper pair, f-wave ($\ell = 3$); heavy fermion superconductor UPt_3 is believed to be a triplet f-wave superconductor

Technically, BCS theory can accommodate other pairing symmetries other than s-wave, but in practice, there are no known non-s-wave superconductors that everyone agrees are BCS. As such, non-s-wave pairing symmetry is often sufficient condition to be 'unconventional'. The word "pairing symmetry" sometimes means s-,p-,d-wave, etc or it can refer to a specific sub-category, like $d_{x^2-y^2}$

A momentum space representation of $d_{x^2-y^2}$ superconductivity, as found in cuprates, is shown here:



Here, the big circle represents the Fermi surface. The radial distance from the circle to the edge of the clover leaf at each point represents the magnitude of the superconducting gap at each point, and the magnitude of the gap is different in different directions. The colors represent the **phase** of the superconducting order parameter. Lets say that the blue is $\theta_b = \theta_1$ and the cyan is $\theta_c = \theta_1 + \pi$

Note that this phase angle appears in a complex exponential which can be expressed in terms of sines and cosines

$$\begin{aligned}\psi_c = \psi_0 e^{i(\theta_1+\pi)} &= \psi_0[\cos(\theta_1+\pi) + i\sin(\theta_1+\pi)] \\ &= \psi_0[-\cos(\theta_1) - i\sin(\theta_1)] = -\psi_0 e^{i\theta_1} \\ &= -\psi_b\end{aligned}$$

The point on the Fermi surface where you go from cyan to blue is called the "node". At this point, the **magnitude** of the superconducting gap is identically zero, because it needs to switch sign (see equation above). The antinode is where the superconducting gap is maximum. If this maximum magnitude of the superconducting gap is Δ_0 and the center of the Fermi surface is at the corner of a 2D square BZ (π, π), the superconducting gap (to lowest order) can be expressed as: $\Delta(\boldsymbol{k}) = \Delta(k_x, k_y) \approx \frac{1}{2}(\cos(k_x) - \cos(k_y))$ where k_x and k_y are different crystal momenta along the Fermi surface. This is just one among many examples of how a superconducting gap can vary in momentum space.

Most experiments are only sensitive to the magnitude of the superconducting gap. For example, heat capacity at low temperature will still tell you about the gap magnitude but for a $d_{x^2-y^2}$ gap (or any superconducting gap with nodes), there **will** be zero energy excitations, such that the temperature dependence will not be exponential; different types of nodes in the gap structure will have different thermodynamic signatures. Angle-resolved photoemission spectroscopy can measure the magnitude of the superconducting gap as a function of momentum directly, as long as the Tc is not too low and the gap is not too small.

One experiment that can measure phases is specially constructed SQUID devices. A sketch is shown below, from the following review article:
https://journals.aps.org/rmp/abstract/10.1103/RevModPhys.67.515

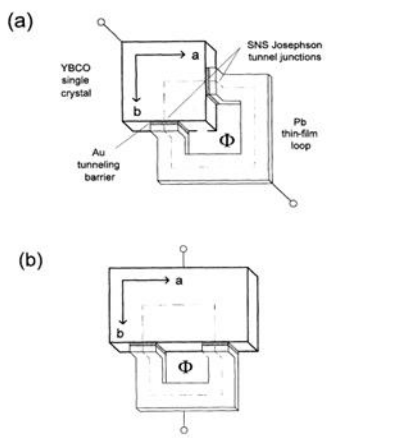
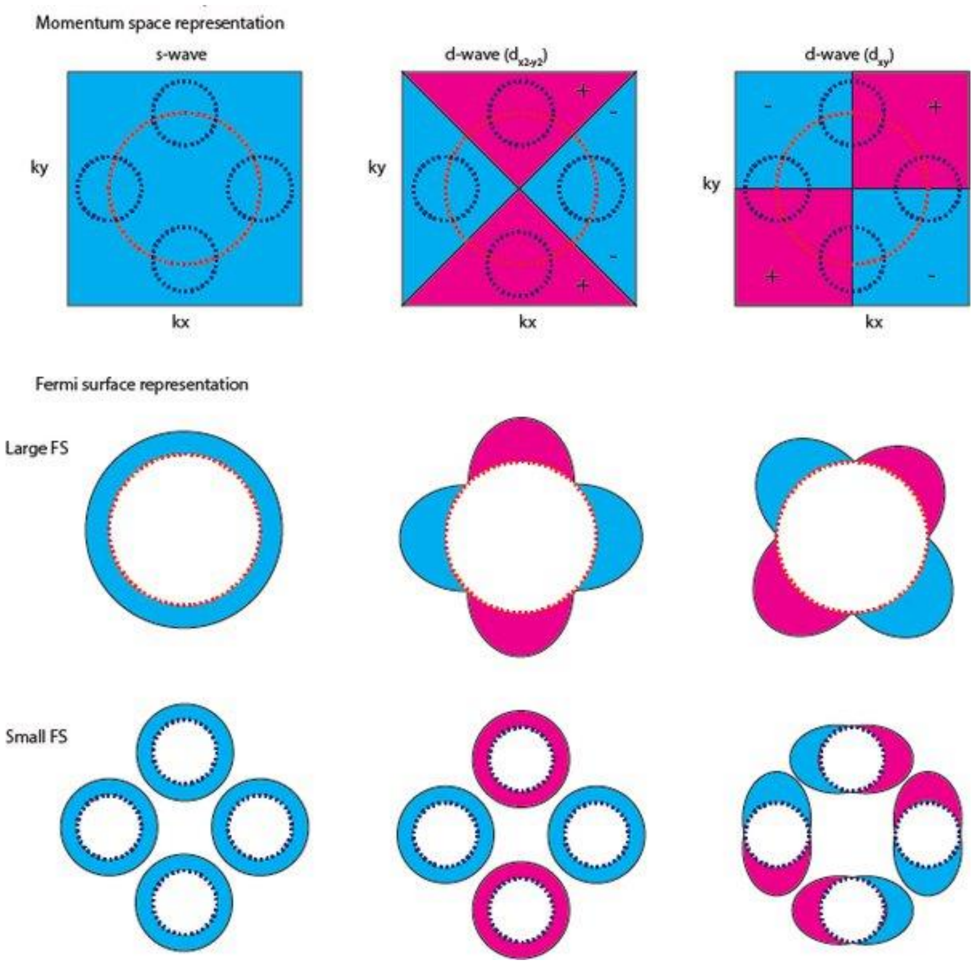


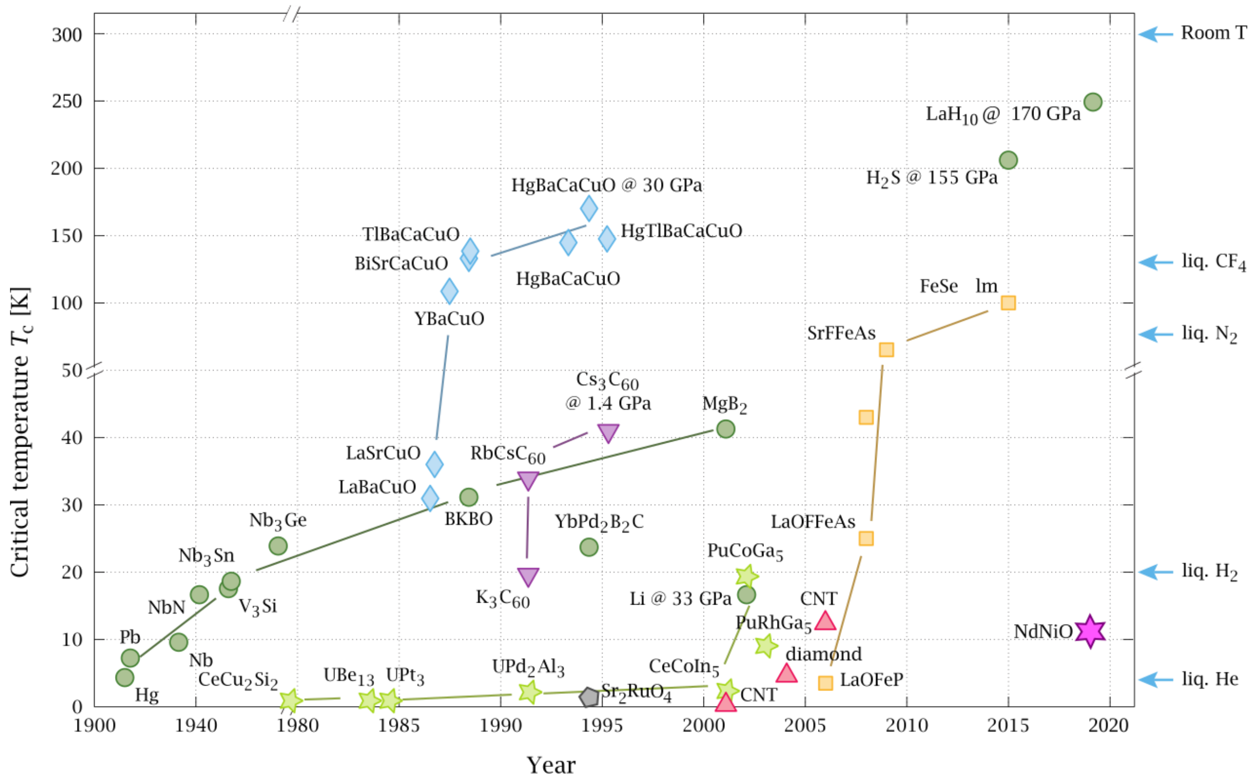
FIG. 4. Design of the SQUID experiments: (a) Configuration of the corner SQUID interferometer experiment used to determine the relative phase between orthogonal directions. (b) Configuration of the edge SQUIDs used as a control sample, in which both junctions are on the same crystal face.

Note that d-wave pairing symmetry is not necessarily synonymous with nodes in the gap structure; it also depends on what the Fermi surface looks like and where they are in the Brillouin zone. The image below contrasts s-wave (for simplicity, say it always had isotropic gap) with two types of d-wave symmetries; two types of Fermiologies are included: one large Fermi surface and 4 small Fermi surfaces.

If the small Fermi surface is fully inside a sector with one phase (i.e. fully inside blue or fully inside cyan), it will not show a node, even if d-wave. Additionally, different types of d-wave symmetries can put nodes in different places in momentum space (and different orientations relative to crystallographic axes.



Below is a history of superconductivity graph, often shown at the start of talks. I will say a few words about most of the characters on there. Image source: “By PJRay - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=46193149”



- Green circles: conventional superconductors. Mechanism explained by BCS theory. Note that hydrogen-containing versions can have very high Tc at high pressure, demonstrating that BCS is not synonymous with low Tc.
- Green stars: Heavy fermions, called such because electrons behave as if they have large effective mass in normal state. Superconductivity often found proximate to antiferromagnetism, prompting theories involving magnetic-fluctuation mediated pairing. However, pairing mechanism is still not resolved.
- Blue diamonds: cuprate high temperature superconductors. Mechanism not resolved. Superconductivity found proximate to various phases: antiferromagnetic insulator, CDW correlations, mystery phase called pseudogap. Only compounds to superconduct above liquid nitrogen temperature.
- Purple and pink triangles: various types of carbon-based superconductors
- Orange squares: iron-based high temperature superconductors; mechanism not resolved. Superconductivity found proximate to magnetic phases; nematic order/correlations thought to be important. A lot of research (at the time of writing these notes) focuses on potential instances of topological superconductivity. Highest Tc data point has not been reproduced.
- Cyan star: nickelates (this graph has not been updated; higher Tc has been seen under pressure); long sought after analog of cuprates, where SC shows up in layered transition metal oxide.